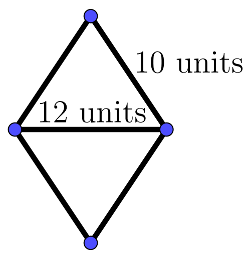
THE FORTY FIRST W.J. BLUNDON MATHEMATICS CONTEST

Sponsored by
The Canadian Mathematical Society *
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 26, 2025

1. Find the area of a rhombus with a side length 10 units and the shortest diagonal 12 units.



Solution: Draw the second diagonal of the rhombus. Observe that in any rhombus diagonals intersect at the right angle and divide each other in halves. Thus, we obtain four identical right triangles with sides $6, 10$, and $\sqrt{10^2 - 6^2} = 8$ units. The area of one such triangle is $\frac{6\times8}{2} = 24$ square units. So, the area of the rhombus is $4 \times 24 = 96$ square units.

The answer is 96 square units.

2. Consider the number consisting of 60 digits

101112131415161718192021222324252627282930313233343536373839

What is the biggest 6-digit number that can be obtained by erasing 54 digits of your choice?

Note: remaining digits should be in the same order as in the original number.

Solution:

$$\underbrace{1011121314151617181}_{\textit{erase 19 digits}}9\underbrace{2021222324252627282}_{\textit{erase 19 digits}}9\underbrace{303132333435363}_{\textit{erase 15 digits}}7\underbrace{\ 3\ }_{\textit{erase}}839$$

Answer: 997839.

3. A closed plastic rectangular box is partially filled with 160 m^3 of water. The depth of the water is either 2 m or 4 m or 5 m, depending on which face of the box is on the ground. What is the volume of the box?

Solution: Let a, b, c be the the side lengths (in meters) of the box.

We have $ab = 160/2 = 80$, $bc = 160/4 = 40$, $ac = 160/5 = 32$.

Then $(abc)^2 = 80 \times 40 \times 32 = 32 \times 32 \times 100$, so the volume is $abc = \sqrt{32 \times 32 \times 100} = 320$.

The answer is 320 m^3.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

4. Three different numbers are chosen at random from the set $\{1, 2, 3, \ldots, 9\}$. What is the probability that one of them is the average of the other two?

   Solution: There are $C_9^3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7}{3\times2} = 12 \times 7 = 84$ ways to select 3 numbers from the set of nine. Without loss of generality we can write each triple of numbers in the increasing order.

   Out of all possible ordered triples there are 16 triples for which the middle number is the average of the other two:

   123, 135, 147, 159, 234, 246, 258, 345, 357, 369, 456, 568, 567, 579, 678, 789.

   The probability is found as the ratio of the number of favourable events to the number of all possible events: $\frac{16}{84} = \frac{4}{21}$.

   The answers is $\frac{4}{21}$.

5. (a) Find the integer part of the number $\sqrt{42+\sqrt{42+\sqrt{42+\sqrt{42+\sqrt{42+\sqrt{42}}}}}}$.

   (b) Find the integer part of the number $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}}}$,

   where $a = n(n+1)$ for some positive integer number n. Explain your result.

   Solution:

   (a) Observe that $6 < \sqrt{42} < 7$. Then $6 < \sqrt{42+\sqrt{42}} < 7$, and so on. The answer is 6.

   (b) Let $x = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a....}}}}}}$ (an infinite nested root). Then $x = \sqrt{a+x}$,

   so $x^2 = x + a$ and $x^2 - x - n(n+1) = 0$. This equation has a positive root $x = n+1$.

   Observe that the sequence $x_k = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a....}}}}}}$ (k-th nested root) is increasing and the limit (as k approaches the infinity) of this sequence is x. Also, $x_1 = \sqrt{n(n+1)} > n$. Therefore, the integer part of every term is $n < x_k < n+1$.

   The answer is n.

6. (a) What is the smallest angle that is formed by the hour and the minute hands at 5:00?

   Answer: 150 deg (The hour hand moves 30 deg per hour.)

   (b) What is the smallest angle that is formed by the hour and the minute hands at 11:10?

   Answer: 85 deg (The hour hand moves 30 deg per hour = 0.5 deg per min. The minute hand moves 6 deg per minute.)

   (c) What is the first time after 1:00 when the two hands form an angle of 80 deg?

   Answer: 1:20

*

A grant in support of this activity was received from the Canadian Mathematical Society.
La Société mathématique du Canada a donné un appui financier à cette activité.

(d) How many times during any 24 hour period do the minute hand and the hour hand form an angle of 90 degrees?

Answer: 44 (Twice between every N:00 and N:59 for $1 \le N \le 12$ except when $N = 2$ and $N = 8$. In the two exceptional intervals it happens only once).

7. Find all values (if any) of the parameter a for which the equation $10 - |x| = ax$ has

a) exactly one root, which is a positive number. Express this root in terms of a.

b) exactly two distinct roots. Express these roots in terms of a.

c) no roots.

Solution: Let $x \ge 0$, so $|x| = x$. Then $10 - x = ax$ and $x = \frac{10}{a+1}$ for any $a > -1$.

Let $x < 0$, so $|x| = -x$. Then $10 + x = ax$ and $x = \frac{10}{a-1}$ for any $a < 1$.

Therefore the answer is:

a) If $a > 1$ the only root is $x = \frac{10}{a+1}$ (which is a positive number).

b) If $-1 < a < 1$ there are two roots $x = \frac{10}{a+1}$ and $x = \frac{10}{a-1}$.

c) No such values of the parameter a. (The equation always has at least one root. Specifically, if $a < -1$ the only root is $x = \frac{10}{a-1}$, which is a negative number.)

8. The 5×5 square is filled with numbers in such a way that each row and each column contains the numbers 1, 2, 3, 4 and 5 exactly once.

The sum of the numbers in each of the three areas marked with either & or # or $ is equal.

&	&	&	#	#
&	&	#	#	#
&	#	#	#	$
#	#	#	$	$
#	#	$	$	$

If the number in the upper left corner is 3, what are the other numbers (b, c, d, and e) on the diagonal of the square?

3				
	b			
		c		
			d	
				e

Solution: Observe that $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the numbers in each area is $(15 \times 5)/3 = 25$. In the area marked with &, we need five numbers whose sum is 22. Then we need 13 numbers that sum to 25 for the area marked with # and six numbers that sum to 25 for the area marked with $. The following arrangement satisfies all the conditions.

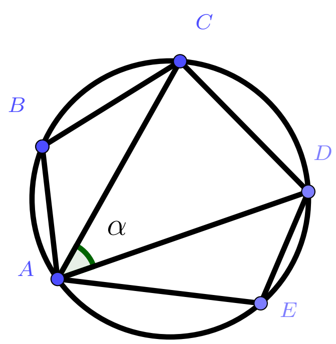
* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

3	5	4	2	1
5	4	2	1	3
4	2	1	3	5
2	1	3	5	4
1	3	5	4	2

The answer is $b = 4$, $c = 1$, $d = 5$, $e = 2$.

9. Five distinct points A, B, C, D, E lie on the same circle in this order. Let angle $\angle CAD = \alpha$. Find the sum of the angles $\angle ABC$ and $\angle AED$ in terms of α.

   Solution:

   Consider inscribed quadrilateral ABCD. We know that $\angle ABC + \angle ADC = 180°$.

   Consider inscribed quadrilateral AEDC. We know that $\angle AED + \angle ACD = 180°$.

   From triangle ACD we know that $\angle ADC + \angle ACD = 180° - \alpha$.

   Therefore, $\angle ABC + \angle AED + 180° - \alpha = 360°$, so $\angle ABC + \angle AED = 180° + \alpha$.

   The answer is $180° + \alpha$.

10. Do there exist positive integers $A > 0$ and $B > 0$ such that both $A^2 + B$ and $B^2 + A$ are perfect squares (squares of some integers)?

    Solution:

    Consider the case $A = B$. Our question becomes: is it possible that $A^2 + A$ is a perfect square?

    Since $A > 0$, the number $A^2 + A > A^2$.

    On the other hand, $(A + 1)^2 = A^2 + 2A + 1 > A^2 + A$.

    Thus, the number $A^2 + A$ is between two consecutive squares, so it cannot be a perfect square.

    Let $A \neq B$. Without loss of generality, assume $A > B$.

    Then $2A > B$ and so $(A + 1)^2 = A^2 + 2A + 1 > A^2 + B$.

    Also, since $B > 0$, we have $A^2 + B > A^2$.

    The number $A^2 + B$ is between two consecutive squares, so it can't be a perfect square.

    There are no such A and B.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.